Controllability of control systems simple Lie groups and the topology of flag manifolds

Ariane Luzia dos Santos[*][†][‡]   Luiz A. B. San Martin[§]

Abstract

Let S be subsemigroup with nonempty interior of a complex simple Lie group G. It is proved that $S = G$ if S contains a subgroup $G(\alpha) \approx$ $\mathrm{Sl}(2, \mathbb{C})$ generated by the $\exp \mathfrak{g}_{\pm \alpha}$, where $\mathfrak{g}_\alpha$ is the root space of the root α. The proof uses the fact, proved before, that the invariant control set of S is contractible in some flag manifold if S is proper, and exploits the fact that several orbits of $G(\alpha)$ are 2-spheres not null homotopic. The result is applied to revisit a controllability theorem and get some improvements.

AMS 2010 subject classification: 93B05, 22E10, 22F30
Key words and phrases: Controllability, simple Lie groups, flag manifolds.

1   Introduction

In this paper we use a method to study controllability of bilinear control systems and invariant control systems on (semi-)simple Lie groups that relies on the (algebraic) topology of flag manifolds.

1

The method is based on the geometry of invariant control sets on flag manifolds, as described initially in [12], [15] and [13], and further developed in [14], [3], [17], [16], [11], [18].

The part of this geometry to be applied here states that if $S \subset G$ is a semigroup with nonempty interior then there exists some flag manifold of G, say $\mathbb{F}_\Theta$, such that the unique invariant control set $C_\Theta \subset \mathbb{F}_\Theta$, for the action of S on $\mathbb{F}_\Theta$, is contained in a subset $\mathcal{E} \subset \mathbb{F}_\Theta$, which is homeomorphic to an Euclidian space $\mathbb{R}^N$ (cf. Theorem 2.4 below). ($\mathcal{E} \approx \mathbb{R}^N$ is an open Bruhat cell of $\mathbb{F}_\Theta$.)

This implies, for instance, that any closed curve γ contained in C_Θ is homotopic (in $\mathbb{F}_\Theta$) to a point, and hence represents a trivial element of the fundamental group $\pi_1 (\mathbb{F}_\Theta)$. Analogously, any higher dimensional sphere $S^n \subset C_\Theta$ represents the identity of the homotopy group $\pi_n (\mathbb{F}_\Theta)$.

Therefore one can achieve to prove $S = G$ by showing that the invariant control sets in all flag manifolds are topologically non trivial e.g. contains a curve or a sphere not homotopic (in the flag manifold) to a point. In particular one gets controllability (in G) of an invariant control system on G by applying this method to the semigroup of control S, as soon as the control system satisfies the Lie algebra rank condition.

In this paper we give sufficient conditions for controllability (under the Lie algebra rank condition) taking advantage of the fact that some subgroups of G have orbits on the flag manifolds that are homeomorphic to spheres but not homotopic to a point. Specifically, we consider subgroups $G(\alpha) \subset G$ where α is a root of the Lie algebra $\mathfrak{g}$ of G and $G(\alpha)$ is generated by (the exponentials of) the root spaces $\mathfrak{g}_\alpha$ and $\mathfrak{g}_{-\alpha}$.

Then our main result (see Theorem 3.1 below) says that in a complex Lie group G the semigroup $S = G$ if int$S \neq \emptyset$ and $G(\alpha) \subset S$ The technique of proof consist in i) checking that several orbits $G(\alpha) \cdot x$ are 2-spheres not homotopic to a point; and ii) some of these orbits are contained in the unique invariant control set C_Θ of S. If this is done in any flag manifold $\mathbb{F}_\Theta$ then no C_Θ is contained in a contractible subset, and S must be G.

Our source of inspiration to think in the group $G(\alpha)$ is a series of papers started with Jurdjevic-Kupka [8], [9], followed by several papers (see Gauthier-Kupka-Sallet [6] and references therein), and culminating with the final result of El Assoudi-Gauthier-Kupka [1]. One of the main issues in these papers is that the semigroup of control S contains a regular element as well as $G(\mu)$ when μ is the highest root.

Thus our Theorem 3.1 provides an alternate proof of the main theorem

2

of [1], when the group G is simple and complex. Actually, we improve that result for these groups. This is because our result is for an arbitrary root α, and not just the highest one. We state this improvement in Theorem 4.2.

We work with simple groups to avoid to take all the time the decomposition into simple components, which can be done in the standard fashion, and is left to the reader.

Similar results can be obtained for real simple groups although the topology of their flag manifolds is trickier. We leave to a forthcoming paper the case of the so-called normal real forms where all the roots have multiplicity one, and hence the orbits $G(\alpha) \cdot b_\Theta$ have dimension zero or one. In this case we must look at the fundamental groups $\pi_1(\mathbb{F}_\Theta)$.

2 Semigroups and flag manifolds

For a complex semi-simple Lie group G with Lie algebra $\mathfrak{g}$ we use the following notation:

- $\mathfrak{h}$ is a Cartan subalgebra, whose set of roots is denoted by Π. Π^+ is a set of positive roots with

$$\Sigma = \{\alpha_1, \ldots, \alpha_l\} \subset \Pi^+$$

  standing for the corresponding simple system of roots. We have $\Pi = \Pi^+ \dot\cup (-\Pi^+)$ and any $\alpha \in \Pi^+$, is a linear combination $\alpha = n_1\alpha_1 + \cdots + n_l\alpha_l$ with $n_i \geq 0$ integers. The support of α, $\operatorname{supp}\alpha$ is the subset of Σ where $n_i > 0$.

- The Cartan-Killing form of $\mathfrak{g}$ is denoted by $\langle \cdot, \cdot \rangle$. If $\alpha \in \mathfrak{h}^*$ then $H_\alpha \in \mathfrak{h}$ is defined by $\alpha(\cdot) = \langle H_\alpha, \cdot \rangle$, and $\langle \alpha, \beta \rangle = \langle H_\alpha, H_\beta \rangle$. The subspace spanned over $\mathbb{R}$ by H_α, $\alpha \in \Pi$, is denoted by $\mathfrak{h}_\mathbb{R}$. We have $\mathfrak{h} = \mathfrak{h}_\mathbb{R} + i\mathfrak{h}_\mathbb{R}$.

- We write
$$\mathfrak{h}_\mathbb{R}^+ = \{H \in \mathfrak{h}_\mathbb{R} : \forall \alpha \in \Pi^+, \, \alpha(H) > 0\}$$
  for the Weyl chamber defined by Π^+.

- The root space of a root α is

$$\mathfrak{g}_\alpha = \{X \in \mathfrak{g} : \forall H \in \mathfrak{h}, \, [H, X] = \alpha(H)X\}.$$

  It is known that $\dim_\mathbb{C} \mathfrak{g}_\alpha = 1$.

3

- For a root α, $\mathfrak{g}(\alpha)$ is the subalgebra generated by $\mathfrak{g}_\alpha$ and $\mathfrak{g}_{-\alpha}$. Then

$$\mathfrak{g}(\alpha) = \operatorname{span}_{\mathbb{C}}\{H\} \oplus \mathfrak{g}_\alpha \oplus \mathfrak{g}_{-\alpha} \approx \mathfrak{sl}(2, \mathbb{C}).$$

  $G(\alpha)$ is the connected Lie subgroup with Lie algebra $\mathfrak{g}(\alpha)$, which is isomorphic to $\operatorname{Sl}(2, \mathbb{C})/D$, where D is a discrete central subgroup (because $\operatorname{Sl}(2, \mathbb{C})$ is simply connected).

- $\mathfrak{u}$ is a compact real form of $\mathfrak{g}$ and $U = \langle \exp \mathfrak{u} \rangle$ is the connected Lie subgroup with Lie algebra $\mathfrak{u}$. It is known that U is compact semi-simple, and maximal compact in G.

- $\mathcal{W}$ is the Weyl group. Either $\mathcal{W}$ is the group generated by the reflections r_α, $\alpha \in \Pi$, $r_\alpha(\beta) = \beta - \frac{2\langle \alpha, \beta \rangle}{\langle \alpha, \alpha \rangle}\alpha$, or $\mathcal{W} = \operatorname{Norm}_U(\mathfrak{h})/T$ where T is the torus $U \cap \exp \mathfrak{h}$ and $\operatorname{Norm}_U(\mathfrak{h}) = \{g \in U : \operatorname{Ad}(g)\mathfrak{h} \subset \mathfrak{h}\}$ is the normalizer of $\mathfrak{h}$ in U.

- $\mathfrak{n}^+ = \sum_{\alpha \in \Pi^+} \mathfrak{g}_\alpha$ and $\mathfrak{n}^- = \sum_{\alpha \in \Pi^+} \mathfrak{g}_{-\alpha}$

- Given the data $\mathfrak{h}$ and Π^+ (or Σ) there is the Borel subalgebra (minimal parabolic) $\mathfrak{p} = \mathfrak{h} \oplus \mathfrak{n}^+$. A subset $\Theta \subset \Sigma$ defines the standard parabolic subalgebra by

$$\mathfrak{p}_\Theta = \mathfrak{p} + \sum_{\alpha \in \langle \Theta \rangle} \mathfrak{g}_\alpha$$

  where $\langle \Theta \rangle = \{\alpha \in \Pi : \operatorname{supp}\alpha \subset \Theta \text{ or } \operatorname{supp}(-\alpha) \subset \Theta\}$ is the set of roots spanned by Θ. ($\mathfrak{p}_\emptyset = \mathfrak{p}$.)

- For $\Theta \subset \Sigma$, P_Θ is the parabolic subgroup with Lie algebra $\mathfrak{p}_\Theta$, which is the normalizer of $\mathfrak{p}_\Theta$:

$$P_\Theta = \operatorname{Norm}_U(\mathfrak{p}_\Theta) = \{g \in U : \operatorname{Ad}(g)\mathfrak{p}_\Theta \subset \mathfrak{p}_\Theta\}.$$

- The flag manifold $\mathbb{F}_\Theta = G/P_\Theta$, which is independent of the specific group G with Lie algebra $\mathfrak{g}$. The origin of G/P_Θ (the coset $1 \cdot P_\Theta$) is denoted by b_Θ.

Now let $S \subset G$ be a subsemigroup with $\operatorname{int} S \neq \emptyset$. We recall here some results of [12], [13] and [15] that are on the basis of our topological approach to controllability in G.

4

We let S act on a flag manifold $\mathbb{F}_\Theta$, by restriction of the action of G. An invariant control set for S in $\mathbb{F}_\Theta$ is a subset $C \subset \mathbb{F}_\Theta$ such that $\mathrm{cl}\,(Sx) = C$ for every $x \in C$, where $Sx = \{gx \in \mathbb{F}_\Theta : g \in S\}$. Since $\mathrm{int}\,S \neq \emptyset$ such a set is closed, has nonempty interior and is in fact invariant ($gx \in C$ if $g \in S$ and $x \in C$).

Lemma 2.1 *(See [12].) In any flag manifold $\mathbb{F}_\Theta$ there is a unique invariant control set for S, denoted by C_Θ.*

To state the geometrical property of C_Θ to be used later we discuss the dynamics of the vector fields $\widetilde{H}$ on a flag manifold $\mathbb{F}_\Theta$ whose flow is $\exp tH$, with H in the closure $\mathrm{cl}\mathfrak{h}_\mathbb{R}^+$ of Weyl chamber $\mathfrak{h}_\mathbb{R}^+$. It is known that $\widetilde{H}$ is a gradient vector field with respect to some Riemmannian metric on $\mathbb{F}_\Theta$ (see Duistermat-Kolk-Varadarajan [4] and Ferraiol-Patrão-Seco [5]).

Hence the orbits of $\widetilde{H}$ are either fixed points or trajectories flowing between fixed point sets. Moreover, $\widetilde{H}$ has a unique attractor fixed point set, say $\mathrm{att}_\Theta\,(H)$, that has an open and dense stable manifold $\sigma_\Theta\,(H)$ (see [4] and [5]). This means that if $x \in \sigma_\Theta\,(H)$ then its ω-limit set $\omega\,(x)$ is contained in $\mathrm{att}_\Theta\,(H)$. This attractor has the following algebraic expressions

$$\mathrm{att}_\Theta\,(H) = Z_H \cdot b_\Theta = U_H \cdot b_\Theta$$

(see [4] and [5]). Here $Z_H = \{g \in G : \mathrm{Ad}\,(g)\,H = H\}$ is the centralizer of H in G and $U_H = Z_H \cap U$ is the centralizer in U. Its stable set $\sigma_\Theta\,(H)$ is also described algebraically by

$$\sigma_\Theta\,(H) = N_H^- Z_H \cdot b_\Theta$$

where $N_H^- = \exp \mathfrak{n}_H^-$ and

$$\mathfrak{n}_H^- = \sum_{\gamma(H)<0} \mathfrak{g}_\gamma.$$

In particular if H is regular, that is, $H \in \mathfrak{h}_\mathbb{R}$ and $\alpha\,(H) > 0$ for $\alpha \in \Pi^+$ then Z_H reduces to the Cartan subgroup $\exp \mathfrak{h}$, which fixes b_Θ. Hence

$$\mathrm{att}_\Theta\,(H) = Z_H \cdot b_\Theta = \{b_\Theta\} \qquad H \in \mathfrak{h}_\mathbb{R}.$$

Actually, in the regular case the fixed points are isolated because $\widetilde{H}$ is the gradient of a Morse function, see [4] and [5]. Also, $\mathfrak{n}_H^- = \mathfrak{n}^-$ (notation as above) and the stable set is $N^- \cdot b_\Theta$ (open Bruhat cell).

The following statement is a well known result from the Bruhat decomposition of the flag manifolds (see [4], [10], [20]).

Proposition 2.2 *In any flag manifold* $\mathbb{F}_\Theta$ *the open Bruhat cell* $N^- \cdot b_\Theta$ *is diffeomorphic to an Euclidian space* $\mathbb{R}^d$. *(The diffeomorphism is* $X \in \mathfrak{n}_\Theta^- \mapsto \exp X \cdot b_\Theta$, *where* $\mathfrak{n}_\Theta^- = \sum \{\mathfrak{g}_\alpha : \alpha < 0 \text{ and } \alpha \notin \langle \Theta \rangle \}$.*)*

Put $h = \exp H$, $H \in \mathfrak{h}_\mathbb{R}^+$. It follows from the gradient property of $\widetilde{H}$ that $\lim_{n \to +\infty} h^n x = b_\Theta$ for any $x \in N^- \cdot b_\Theta$.

Now, we say that $g \in G$ is **regular real** if it is a conjugate $g = aha^{-1}$ of $h = \exp H$, $H \in \mathfrak{h}_\mathbb{R}^+$ with $a \in G$. Then we write $\sigma_\Theta(g) = g \cdot \sigma_\Theta(H)$ and call this the stable set of g in $\mathbb{F}_\Theta$. (The reason for this name is clear: $g^n = (aha^{-1})^n = ah^n a^{-1}$ and hence $g^n x \to g b_\Theta$ if $x \in \sigma_\Theta(g)$.)

The following lemma was used in [12] to prove the above Lemma 2.1.

Lemma 2.3 *(See [12].) There exists regular real* $g \in \mathrm{int} S$.

Now we can state the next theorem from [15], which is in the basis of our approach to controllability.

Theorem 2.4 *Suppose that* $S \neq G$. *Then there exists a flag manifold* $\mathbb{F}_\Theta$ *such that the invariant control set* $C_\Theta \subset \sigma_\Theta(g)$ *for every regular real* $g \in \mathrm{int} S$.

Corollary 2.5 *If* $S \neq G$ *then* C_Θ *is contained in a subset* $\mathcal{E}_\Theta \subset \mathbb{F}_\Theta$, *which is diffeomorphic to an Euclidian space.*

Remark: It can be proved that there exists a minimal Θ_S satisfying the condition of Theorem 2.4. This Θ_S (or rather the flag manifold $\mathbb{F}_{\Theta_S}$) is called the flag type of S or the parabolic type of S (because of the parabolic subgroup P_{Θ_S}). Several properties of S are derived from this flag type (e.g. the homotopy type of S as in [18] or the connected components of S as in [11]).

3 Root spaces and semigroups

Recall the notation $\mathfrak{g}(\alpha) = \mathrm{span}_\mathbb{C}\{H_\alpha\} \oplus \mathfrak{g}_\alpha \oplus \mathfrak{g}_{-\alpha} \approx \mathfrak{sl}(2, \mathbb{C})$ and $G(\alpha) = \langle \exp \mathfrak{g}(\alpha) \rangle$.

Theorem 3.1 *Let* G *be a simple complex Lie group and* $S \subset G$ *a semigroup with* $\mathrm{int} S \neq \emptyset$. *Then* $S = G$ *if there is a root* α *with* $G(\alpha) \subset S$.

For the proof of this theorem we exploit the fact that $\mathfrak{g}(\alpha)$ is isomorphic to $\mathfrak{sl}(2,\mathbb{C})$, and hence the unique flag manifold of $G(\alpha)$ is a 2-sphere. With this in mind we prove that in any flag manifold $\mathbb{F}_\Theta$ of G there are several $G(\alpha)$-orbits that are 2-spheres. Then we ensure that some of these orbits are contained in the invariant control set C_Θ of S in $\mathbb{F}_\Theta$. Finally we use De Rham cohomology of $\mathbb{F}_\Theta$ to prove that such orbits (2-spheres) are not homotopic to a point. Hence C_Θ is not contained in a contractible set and the theorem follows by Corollary 2.5.

The first step is the following lemma which reduces the proof to some specific roots.

Lemma 3.2 *Let γ and β be roots such that $\beta = w\gamma$ with $w \in \mathcal{W}$. If Theorem 3.1 holds for γ then it is true for β as well.*

Proof: Take a representative $\overline{w}$ of w in the normalizer M^* of $\mathfrak{h}$ in U. Then $\beta = w\gamma$ entails that $G(\beta) = \overline{w}G(\gamma)\overline{w}^{-1}$. Now if $G(\beta) \subset S$ then $G(\gamma)$ is contained in the $\overline{w}S\overline{w}^{-1}$ that has nonempty interior. Hence $\overline{w}S\overline{w}^{-1} = G$ implying that $S = G$. $\qquad\square$

The assumption that G is simple ensures the following fact about the action Weyl group $\mathcal{W}$ on the set of roots Π: It is transitive for the Dynkin diagrams A_l, D_l, E_6, E_7 and E_8, that have only simple edges. For the other diagrams B_l, C_l, F_4 and G_2, there are two orbits which are given by the long and the short roots, respectively.

On the other hand $\mathcal{W}$ acts transitively on the set of chambers. Hence for any $\alpha \in \Pi$ there exists a unique root μ such that $H_\mu \in \mathrm{cl}\mathfrak{h}_\mathbb{R}^+$. Hence there is either one or two roots with $H_\mu \in \mathrm{cl}\mathfrak{h}_\mathbb{R}^+$. By the above lemma it is enough to prove the theorem for these roots.

For the simply laced diagrams or for the long roots in the other diagrams the root μ with $H_\mu \in \mathrm{cl}\mathfrak{h}_\mathbb{R}^+$ is the highest root. In any case we have the following property.

Proposition 3.3 *If μ is a root with $H_\mu \in \mathrm{cl}\mathfrak{h}_\mathbb{R}^+$ then $\mathrm{supp}\mu = \Sigma$.*

Proof: If $\{H_1, \ldots, H_l\}$ is the dual basis of $\Sigma = \{\alpha_1, \ldots, \alpha_l\}$ then $\alpha = \alpha(H_1)\alpha_1 + \cdots + \alpha(H_l)\alpha_l$ for any $\alpha \in \mathfrak{h}^*$. The closure $\mathrm{cl}\mathfrak{h}_\mathbb{R}^+$ of the Weyl chamber is a polyhedral cone spanned by $H_1, \ldots, H_l$. It is known that $\langle A, B \rangle > 0$ for nonzero $A, B \in \mathrm{cl}\mathfrak{h}_\mathbb{R}^+$. Therefore, if $H_\mu \in \mathrm{cl}\mathfrak{h}_\mathbb{R}^+$ then the coefficientes $\mu(H_i)$

satisfy $\mu(H_i) = \langle H_\mu, H_i \rangle > 0$, $i = 1, \ldots, l$, which means that $\mathrm{supp}\mu = \Sigma$. $\square$

From now on μ stands for one of the roots with $H_\mu \in \mathrm{cl}\mathfrak{h}_{\mathbb{R}}^+$.

For the next step we recall the notation of the last section where $\mathrm{att}_\Theta(H_\mu)$ is the attractor fixed point set of $\widetilde{H}_\mu \in \mathrm{cl}\mathfrak{h}_{\mathbb{R}}^+$ with $\sigma_\Theta(H_\mu)$ the stable set.

Now let C_Θ be the (unique) invariant control set of S on $\mathbb{F}_\Theta$. It is closed, S-invariant and has nonempty interior. Hence, it meets the dense set $\sigma_\Theta(H_\mu)$.

Lemma 3.4 *If $G(\mu) \subset S$ then $C_\Theta \cap \mathrm{att}_\Theta(H_\mu) \neq \emptyset$.*

Proof: We have $C_\Theta \cap \sigma_\Theta(H_\mu) \neq \emptyset$ and if $x \in C_\Theta \cap \sigma_\Theta(H_\mu)$ then its ω-limit $\omega(x)$ (w.r.t. $\widetilde{H}_\mu$) is contained in C_Θ, because $\{\exp tH_\mu : t \in \mathbb{R}\} \subset G(\mu) \subset S$. Since $\omega(x) \subset \mathrm{att}_\Theta(H_\mu)$, it follows that $\emptyset \neq \omega(x) \subset C_\Theta \cap \mathrm{att}_\Theta(H_\mu)$. $\square$

Now we look at the orbits $G(\mu) \cdot y$ through points $y \in \mathrm{att}_\Theta(H_\mu) = Z_{H_\mu} \cdot b_\Theta$. First for $y = b_\Theta$ we have the following general result.

Lemma 3.5 *Let $\mathbb{F}_\Theta$ be a flag manifold and β a positive root. Then $G(\beta) \cdot b_\Theta$ is either a 2-sphere or reduces to a point. If $\beta \notin \langle \Theta \rangle$ then $\dim G(\beta) \cdot b_\Theta = 2$. In particular, $\dim G(\mu) \cdot b_\Theta = 2$ if $H_\mu \in \mathrm{cl}\mathfrak{h}_{\mathbb{R}}^+$.*

Proof: The point is that the orbit $G(\beta) \cdot b_\Theta$ equals b_Θ or identifies to the only flag manifold of $G(\beta)$ which is the same as the flag manifold of $\mathrm{Sl}(2, \mathbb{C})$ (because $\mathfrak{g}(\beta) \approx \mathfrak{sl}(2, \mathbb{C})$), which in turn is S^2.

To see this denote by $\mathfrak{g}(\beta)_{b_\Theta}$ the isotropy subalgebra at b_Θ for the action of $G(\beta)$ on $\mathbb{F}_\Theta$. It contains the subalgebra $\mathfrak{p}_\beta = \mathrm{span}\{H_\beta\} \oplus \mathfrak{g}_\beta$ which is a parabolic subalgebra of $\mathfrak{g}(\beta)$. This implies that the isotropy subgroup at b_Θ contains the identity component of the parabolic subgroup $P_\beta = \mathrm{Norm}_{G(\beta)}\mathfrak{p}_\beta \subset G(\beta)$. But any parabolic subgroup of the complex group $G(\beta)$ is connected, hence P_β is contained in the isotropy subgroup at b_Θ, for the action of G. This shows that $G(\beta) \cdot b_\Theta$ is either a 2-sphere or reduces to a point.

Now, if $\beta \notin \langle \Theta \rangle$ then $\mathfrak{g}_{-\beta}$ has zero intersection with the isotropy subalgebra $\mathfrak{p}_\Theta$ at b_Θ (which is the sum of the Cartan subalgra with root spaces). This implies that $\mathfrak{g}(\beta)_{b_\Theta} = \mathfrak{p}_\beta$, and since an istropy subgroup normalizes the isotropy subalgebra, it follows that P_β is exactly the isotropy subgroup at b_Θ for the action of $G(\beta)$. Hence $G(\beta) \cdot b_\Theta \approx G(\beta)/P_\beta \approx S^2$. $\square$

As to the $G\left(\mu\right)$-orbit through $y = g \cdot b_\Theta$, $g \in Z_{H_\mu}$, we write

$$G\left(\mu\right) \cdot y = g\left(g^{-1}G\left(\mu\right)g \cdot b_\Theta\right)$$

so that $G\left(\mu\right) \cdot y$ is diffeomorphic to $g^{-1}G\left(\mu\right)g \cdot b_\Theta$. The Lie group $g^{-1}G\left(\mu\right)g$ has Lie algebra $\mathfrak{g}\left(\mu\right)^g = \operatorname{Ad}\left(g\right)\left(\mathfrak{g}\left(\mu\right)\right)$ also isomorphic to $\mathfrak{sl}\left(2,\mathbb{C}\right)$. Since $\operatorname{Ad}\left(g\right)H_\mu = H_\mu$ there is the root space decomposition

$$\mathfrak{g}\left(\mu\right)^g = \langle H_\mu \rangle \oplus \operatorname{Ad}\left(g\right)\left(\mathfrak{g}_\mu\right) \oplus \operatorname{Ad}\left(g\right)\left(\mathfrak{g}_{-\mu}\right).$$

Lemma 3.6 *Keep the assumption that $H_\mu \in \operatorname{cl}\mathfrak{h}_\mathbb{R}^+$ and take $g \in Z_{H_\mu}$. Then* $\operatorname{Ad}\left(g\right)\left(\mathfrak{g}_\mu\right) \subset \mathfrak{n}^+ = \sum_{\alpha>0} \mathfrak{g}_\alpha$.

Proof: Since $\operatorname{Ad}\left(g\right)H_\mu = H_\mu$, it follows that $\operatorname{Ad}\left(g\right)$ commutes with $\operatorname{ad}\left(H_\mu\right)$ and hence maps the eigenspaces of $\operatorname{ad}\left(H_\mu\right)$ onto themselves. Now $\mathfrak{g}_\mu$ is contained in the $\mu\left(H_\mu\right)$-eigenspace of $\operatorname{ad}\left(H_\mu\right)$. Hence, $\operatorname{Ad}\left(g\right)\mathfrak{g}_\mu$ is contained in the same eigenspace. Now $\mu\left(H_\mu\right) = \langle\mu,\mu\rangle > 0$ and the assumption that $H_\mu \in \mathfrak{h}_\mathbb{R}^+$ implies that the eigenspaces of $\operatorname{ad}\left(H_\mu\right)$ associated to positive eigenvalues are contained in $\mathfrak{n}^+$. Hence $\operatorname{Ad}\left(g\right)\mathfrak{g}_\mu \subset \mathfrak{n}^+$ as claimed. $\qquad\square$

Remark: If μ is the highest root the above lemma has a more precise statement, namely if $g \in Z_{H_\mu}$ then g centralizes $\mathfrak{g}\left(\mu\right)$ and $G\left(\mu\right)$ (see Proposition 3.9 below). Hence $g^{-1}G\left(\mu\right)g = G\left(\mu\right)$ and $G\left(\mu\right) \cdot y = g\left(G\left(\mu\right) \cdot b_\Theta\right)$, what simplifies the proofs to follow.

Lemma 3.7 *Keep the above notation and assumptions. Then $G\left(\mu\right) \cdot y = g\left(g^{-1}G\left(\mu\right)g \cdot b_\Theta\right)$ is either a 2-sphere or reduces to a point.*

Proof: The subalgebra $\mathfrak{p}_\mu = \langle H_\mu \rangle \oplus \operatorname{Ad}\left(g\right)\mathfrak{g}_\mu$ is a parabolic subalgebra of $\operatorname{Ad}\left(g\right)\left(\mathfrak{g}\left(\mu\right)\right)$. Hence as in the proof of Lemma 3.5 it is enough to check that $\mathfrak{p}_\mu$ is contained in the isotropy subalgebra $\mathfrak{g}\left(\beta\right)_{b_\Theta}^g$ (for the action of $g^{-1}G\left(\mu\right)g$) at b_Θ. Clearly $H_\mu \in \mathfrak{g}\left(\beta\right)_{b_\Theta}^g$. On the other hand by the previous lemma $\operatorname{Ad}\left(g\right)\mathfrak{g}_\mu \subset \mathfrak{n}^+$ which in turn is contained in the isotropy subalgebra at b_Θ (for the action of G). Hence $\mathfrak{p}_\mu \subset \mathfrak{g}\left(\mu\right)_{b_\Theta}^g$ and the lemma follows. $\qquad\square$

This proof shows that the orbit $g^{-1}G\left(\mu\right)g \cdot b_\Theta$ induces a map from the flag manifold $G\left(\mu\right)/P_\mu \approx S^2$ into $\mathbb{F}_\Theta$. We denote this map by $\sigma_{g,\mu} : S^2 \to \mathbb{F}_\Theta$.

The next, and final step, is to check that the 2-spheres appearing in the last lemma are not homotopic to a point (at least a great amount of them).

The idea is to exhibit a differential 2-form Ω on $\mathbb{F}_\Theta$ with $d\Omega = 0$ such that the pull-back $\nu = \sigma^*_{g,\mu}\Omega$ is a (non zero) volume form on S^2. This would prove that the map $\sigma^*_{g,\mu} : H^2(\mathbb{F}_\Theta, \mathbb{R}) \to H^2(S^2, \mathbb{R})$ induced on cohomology by $\sigma_{g,\mu}$ is not trivial, implying that $\sigma_{g,\mu}$ is not homotopic to a constant map.

In fact, a volume form ν on the orientable manifold S^2 is a generator of its 1-dimensional de Rham cohomology $H^2(S^2, \mathbb{R})$, that is, $d\nu = 0$ and ν is not $d\eta$ for a 1-form η. If $\nu = \sigma^*_{g,\mu}\Omega$ then Ω is not exact, for otherwise $\Omega = d\omega$ imply $\nu = \sigma^*_\mu d\omega = d\sigma^*_\mu \omega$ and ν would be exact as well. Hence Ω represents a non zero element in the De Rham cohomology $H^2(\mathbb{F}_\Theta, \mathbb{R})$ and the image of its cohomology class under $\sigma^*_{g,\mu}$ is the cohomology class $[\nu] \neq 0$.

A 2-form Ω that does the job is a U-invariant symplectic form associated to an invariant Hermitian metric together with a complex structure on $\mathbb{F}_\Theta$ (Kähler form). The construction of these geometric objects goes back to Borel [2]. To define it we follow [19]. First we need a special basis of the tangent space $T_{b_\Theta}\mathbb{F}_\Theta$ at the origin. To get it start with a Weyl basis of $\mathfrak{g}$ which is given by the choice of a generator X_α of the root space $\mathfrak{g}_\alpha$, for each root α and satisfying the conditions $\langle X_\alpha, X_{-\alpha} \rangle = 1$ and $[X_\alpha, X_\beta] = m_{\alpha,\beta}X_{\alpha+\beta}$ with $m_{\alpha,\beta} \in \mathbb{R}$ (see [19], for details). Then define $A_\alpha = X_\alpha - X_{-\alpha}$, $S_\alpha = i(X_\alpha + X_{-\alpha})$ and $\mathfrak{u}_\alpha = \mathrm{span}\{A_\alpha, S_\alpha\}$ with α a positive roots. Both A_α and S_α belong to the compact real form $\mathfrak{u}$ of $\mathfrak{g}$ (Lie algebra of U). By the action of U on $\mathbb{F}_\Theta$ we get the induced vector fields $\widetilde{A}_\alpha$ and $\widetilde{S}_\alpha$. Then we have

- the set $\widetilde{\mathfrak{u}}_\alpha = \{\widetilde{A}_\alpha(b_\Theta), \widetilde{S}_\alpha(b_\Theta) : \alpha \notin \langle\Theta\rangle\}$ is a basis of $T_{b_\Theta}\mathbb{F}_\Theta$.

Now, Ω is defined by specifying its value Ω_0 at the origin and then extending to the whole $\mathbb{F}_\Theta$ by the action of U. The extension is possible if Ω_0 is invariant by the isotropy representation of U_Θ on $T_{b_\Theta}\mathbb{F}_\Theta$.

To get Ω_0 we choose first real numbers $\lambda_\alpha > 0$, $\alpha \in \Pi^+$, satisfying

1. $\lambda_{\alpha+\beta} = \lambda_\alpha + \lambda_\beta$ if α, β and $\alpha + \beta$ are positive roots.

2. $\lambda_{\alpha+\gamma} = \lambda_\alpha$ if α, γ and $\alpha + \gamma$ are roots with $\gamma \in \langle\Theta\rangle$.

Remark: Although it will not be used below we note that the numbers $\lambda_\alpha > 0$, $\alpha \in \Pi^+$, define an inner product on $T_{b_\Theta}\mathbb{F}_\Theta$, which by the second condition is U_Θ-invariant, and hence extends to a Riemmannian metric g on

$\mathbb{F}_\Theta$. The first condition ensures that a Hermitian metric built from g and a complex structure J on $\mathbb{F}_\Theta$ has a Kähler form which is symplectic (see [19], Section 2.4).

Now we define Ω_0 by declaring that $\Omega_0(X, Y) = 0$ if $X = \widetilde{\mathfrak{u}}_\alpha$ and $Y = \widetilde{\mathfrak{u}}$ with $\alpha \neq \beta$ and

$$\Omega_0\left(\widetilde{A}_\alpha(b_\Theta), \widetilde{S}_\alpha(b_\Theta)\right) = \lambda_\alpha \qquad \alpha \in \Pi^+.$$

The second condition above ensures that Ω_0 is a 2-form on $T_{b_\Theta}\mathbb{F}_\Theta$ invariant by U_Θ, and hence defines a 2-form Ω on $\mathbb{F}_\Theta$, by translation. On the other hand from the first condition we have $d\Omega = 0$ (see [19], Proposition 2.1).

Now we are prepared to prove that the 2-spheres are not homotopic to a point.

Lemma 3.8 *Let μ be a positive root such that $H_\mu \in \mathfrak{h}_\mathbb{R}^+$, and denote by Z_{H_μ} the centralizer of H_μ in G, and put $U_{H_\mu} = Z_{H_\mu} \cap U$. Take a flag manifold $\mathbb{F}_\Theta$. Then there exists a subset $V \subset \mathrm{att}_\Theta(H_\mu) = U_{H_\mu} \cdot b_\Theta$ open and dense in $\mathrm{att}_\Theta(H_\mu)$ such that for every $x \in V$, the orbit $G(\mu) \cdot x$ is a 2-sphere not homotopic to a point.*

Proof: For any $x \in \mathrm{att}_\Theta(H_\mu)$ we write $x = g \cdot b_\Theta$ with $g \in U_{H_\mu}$. Since $H_\mu \in \mathfrak{h}_\mathbb{R}^+$, we have by Proposition 3.3 that $\mathrm{supp}\mu = \Sigma$, so that by Lemma 3.5, $G(\mu) \cdot b_\Theta$ is a 2-sphere. Its tangent space $T_{b_\Theta}(G(\mu) \cdot b_\Theta)$ has the basis $\{\widetilde{A}_\mu(b_\Theta), \widetilde{S}_\mu(b_\Theta)\}$. Analogously, the tangent space at b_Θ of $g^{-1}G(\mu)g \cdot b_\Theta$, which we denote simply by T^g, is spanned by $\{\widetilde{A}_\mu^g(b_\Theta), \widetilde{S}_\mu^g(b_\Theta)\}$ where $A_\mu^g = \mathrm{Ad}(g)A_\mu$ and $S_\mu^g = \mathrm{Ad}(g)S_\mu$. Either both vectors $\widetilde{A}_\mu^g(b_\Theta)$ and $\widetilde{S}_\mu^g(b_\Theta)$ are zero or they form a basis of T^g.

Now we pull-back the symplectic form Ω to T^g and define the function $\phi: U_{H_\mu} \cdot b_\Theta \to \mathbb{R}$ by

$$\phi(g \cdot b_\Theta) = \Omega\left(\widetilde{A}_\mu^g(b_\Theta), \widetilde{S}_\mu^g(b_\Theta)\right) \qquad g \in U_{H_\mu},$$

which is well defined because any $g \in U_{H_\mu}$ leaves Ω invariant and $g \cdot b_\Theta = g_1 \cdot b_\Theta$ implies that $\{\widetilde{A}_\mu^g(b_\Theta), \widetilde{S}_\mu^g(b_\Theta)\}$ and $\{\widetilde{A}_\mu^{g_1}(b_\Theta), \widetilde{S}_\mu^{g_1}(b_\Theta)\}$ span the same subspace, namely $T^g = T^{g_1}$. Hence $\Omega\left(\widetilde{A}_\mu^g(b_\Theta), \widetilde{S}_\beta^g(b_\Theta)\right) = \Omega\left(\widetilde{A}_\mu^{g_1}(b_\Theta), \widetilde{S}_\mu^{g_1}(b_\Theta)\right)$

The function ϕ is analytic as is the map $g \mapsto \mathrm{Ad}(g)$. It is not idencally zero, since by Lemma 3.5 we have

$$\phi(b_\Theta) = \Omega\left(\widetilde{A}_\mu(b_\Theta), \widetilde{S}_\mu(b_\Theta)\right) = \lambda_\mu \neq 0.$$

11

Hence the subset $V = \{x \in U_{H_\mu} \cdot b_\Theta : \phi(x) \neq 0\}$ is open and dense in $U_{H_\mu} \cdot b_\Theta$. For any $x \in V$ the orbit $G(\mu) \cdot x$ is 2-dimensional. Also, if $x = g \cdot b_\Theta$, $g \in U_{H_\mu}$, then $\Omega\left(\widetilde{A}_\mu^g(b_\Theta), \widetilde{S}_\mu^g(b_\Theta)\right) \neq 0$. Now Ω is invariant by $(g^{-1}G(\mu)g) \cap U$ and this group acts transitively on $g^{-1}G(\mu)g \cdot b_\Theta$. Hence the pull-back of Ω to $g^{-1}G(\mu)g \cdot b_\Theta$ is a volume form, which shows that $S^2 \approx g^{-1}G(\mu)g \cdot b_\Theta$ is not homotopic to a point if $x = g \cdot b_\Theta \in V$.

In conclusion, we have $G(\mu) \cdot x = g(g^{-1}G(\mu)g \cdot b_\Theta)$, so that $G(\mu) \cdot x$ is not homotopic to a point as well. $\qquad\square$

End of proof of Theorem 3.1: If $x = ng \cdot b_\Theta \in \sigma_\Theta(H_\mu)$ with $n \in N^-$ and $g \in Z_{H_\mu}$ then $(\exp tH_\mu) ng \cdot b_\Theta = (\exp tH_\mu) n (\exp(-tH_\mu)) g \cdot b_\Theta$ because $(\exp(-tH_\mu)) g = g (\exp(-tH_\mu))$ and $(\exp(-tH_\mu)) \cdot b_\Theta = b_\Theta$. However $\lim_{t\to\infty} (\exp tH_\mu) n (\exp(-tH_\mu)) = 1$, so that

$$\lim_{t\to+\infty} (\exp tH_\mu) ng \cdot b_\Theta = g \cdot b_\Theta.$$

Now, let V be the open and dense subset of $Z_{H_\beta} \cdot b_\Theta$ ensured by the last lemma. Then $N_\mu^- \cdot V$ is open and dense in $\mathbb{F}_\Theta$. Since C_Θ has non empty interior we can find $x \in C_\Theta$ such that

$$\lim_{t\to+\infty} (\exp tH) x = y \in C_\Theta \cap V.$$

Then $G(\mu) \cdot y$ is a 2-sphere not homotopic to a point contained in C_Θ. This shows that C_Θ cannot be contained in a contractible subset of $\mathbb{F}_\Theta$. Since Θ was arbitrary $S = G$. In view of Lemma 3.2, this proves Theorem 3.1. $\qquad\square$

To conclude this section we prove the following statement ensuring that for the highest root we have $gG(\mu)g^{-1} = G(\mu)$, $g \in Z_{H_\mu}$, so that the set V of Lemma 3.8 is the totality of $\mathrm{att}_\Theta(H_\mu)$.

Proposition 3.9 *Let μ be the highest root, and suppose that $g \in G$ centralizes H_μ that is $\mathrm{Ad}(g) H_\mu = H_\mu$. Then g normalizes $G(\mu)$ (actually g comutes with every $h \in G(\mu)$).*

Proof: Write Z_{H_μ} for the centralizer of H_μ in G. It is a Lie group with Lie algebra $\mathfrak{z}_{H_\mu} = \ker \mathrm{ad}(H_\mu)$, the centralizer of H_μ in $\mathfrak{g}$. By the root space decomposition we have

$$\mathfrak{z}_{H_\mu} = \mathfrak{h} + \sum_{\beta(H_\mu)=0} \mathfrak{g}_\beta.$$

12

Write $\Theta_{H_\mu} = \{\alpha \in \Sigma : \alpha(H_\mu) = 0\}$. Then a root β anihilates H_μ if and only if supp$\beta \subset \Theta_{H_\mu}$. This follows from the fact that $H_\mu \in \mathrm{cl}\mathfrak{h}_{\mathbb{R}}^+$, so that if $\alpha \notin \Theta_{H_\mu}$ then $\alpha(H_\mu) > 0$.

Take a root β with $\beta(H_\mu) = \langle\beta,\mu\rangle = 0$. Then $\mu \pm \beta$ are not roots. In fact if $\beta > 0$ then $\mu + \beta$ is not a root since $\beta > 0$. Hence by the Killing formula, the orthogonality $\langle\beta,\mu\rangle = 0$ implies that $\mu - \beta$ is neither a root. We have also that $-\mu \pm \beta$ are not roots. Therefore $[\mathfrak{g}_{\pm\mu}, \mathfrak{g}_\beta] = 0$ if $\beta(H_\mu) = 0$, and since $\mathfrak{h}$ is abelian we conclude that $[X, \mathfrak{g}(\mu)] = 0$ if $X \in \mathfrak{z}_{H_\mu}$.

Now, since we are working with the complex group G it is true that Z_{H_μ} is connected. Hence the commutativity between $\mathfrak{z}_{H_\mu}$ and $\mathfrak{g}(\mu)$ implies that $\mathrm{Ad}(g)Y = Y$ for any $Y \in \mathfrak{z}_{H_\mu}$. This in turn implies the elements of Z_{H_μ} commute with the elements of $G(\mu)$. $\square$

Corollary 3.10 *Let μ be the highest root and denote by Z_{H_μ} the centralizer of H_μ in G. Take a flag manifold $\mathbb{F}_\Theta$. Then for any $g \in Z_{H_\mu}$ the orbit $G(\mu) \cdot gb_\Theta$ is a 2-sphere in $\mathbb{F}_\Theta$ not homotopic to a point.*

4   Controllability theorem

As mentioned in the introduction our source of inspiration for Theorem 3.1 are the results on controllability of control systems of [8], [9], [6], [1]. The starting point in the proof of these results is the proof that $G(\mu)$ is contained in the semigroup of control. Their assumptions are designed to ensure this inclusion. With Theorem 3.1 we can improve (for complex Lie groups) the final theorem of [1], without insisting to work with the highest root.

Let

$$\dot{g} = (A + u(t)B)g \qquad u(t) \in \mathbb{R} \tag{1}$$

be a rigth invariant control system with unrestricted controls where $A, B \in \mathfrak{g}$ with $\mathfrak{g}$ a complex Lie algebra of the complex Lie group G. We let S be the semigroup of the system (generated by $\exp t(A + uB)$, $t \geq 0$, $u \in \mathbb{R}$) and denote by

$$\Gamma = \{X \in \mathfrak{g} : \forall t \geq 0, \exp tX \in \mathrm{cl}S\}$$

its Lie wedge. Γ is a closed convex cone invariant by $\exp t\,\mathrm{ad}(X)$, $t \in \mathbb{R}$, if $\pm X \in \Gamma$ (see Hilgert-Hofmann-Lawson [7]).

Since (1) is with unrestricted controls the following easy argument shows that $\pm B \in \Gamma$: $A + uB \in \Gamma$, $u \in \mathbb{R}$, hence if $u > 0$, $(1/u) A + B \in \Gamma$, so that $B = \lim_{u \to +\infty} ((1/u) A + B) \in \Gamma$. Similarly $-B \in \Gamma$, by making $u \to -\infty$.

Now we shall take B in the Cartan subalgebra $\mathfrak{h}$ and write

$$A = A_0 + \sum_{\alpha \in \Pi} A_\alpha$$

for the root space decomposition of A, $A_0 \in \mathfrak{h}$ and $A_\alpha \in \mathfrak{g}_\alpha$.

The Cartan subalgebra $\mathfrak{h}$ decomposes as $\mathfrak{h} = \mathfrak{h}_\mathbb{R} + i\mathfrak{h}_\mathbb{R}$ where $\mathfrak{h}_\mathbb{R}$ is the real subspace where the roots assume real values. If β is a root we have $\beta(H) \in \mathbb{R}$ if $H \in \mathfrak{h}_\mathbb{R}$ and $\beta(H)$ is immaginary if $H \in i\mathfrak{h}_\mathbb{R}$.

In particular we write $B = B_{\mathrm{Re}} + B_{\mathrm{Im}} \in \mathfrak{h}_\mathbb{R} + i\mathfrak{h}_\mathbb{R}$, and state the controllability result separetaly into two cases: 1) $\mathrm{ad}(B)$ has purely imaginary eigenvalues, that is, $B_{\mathrm{Re}} = 0$; 2) $B_{\mathrm{Re}} \neq 0$. The proofs follow almost immediately from our Theorem 3.1 and Lemma 2.3 of [1], whose arguments we reproduce, for the sake of completeness.

Theorem 4.1 *In the control system (1) take $B \in \mathfrak{h}$ and suppose that $\mathrm{ad}(B)$ has purely imaginary eigenvalues ($B_{\mathrm{Re}} = 0$). Then the system is controllable in G if*

1. *A and B generate $\mathfrak{g}$ (Lie algebra rank condition), and*

2. *there exists a root α such that $\alpha(B) \neq 0$, $A_\alpha \neq 0 \neq A_{-\alpha}$ and $\alpha(B) \neq \beta(B)$ for any root $\beta \neq \alpha$ with $A_\beta \neq 0$.*

Proof: We have $\exp t\,\mathrm{ad}(B) A \in \Gamma$ for all $t \in \mathbb{R}$ and

$$\exp t\,\mathrm{ad}(B) A = \sum_{\beta \in \Pi} e^{t\beta(B)} A_\beta.$$

Hence $A_\eta(t) = (1 + \eta \cos \alpha(B)) \exp t\,\mathrm{ad}(B) A \in \Gamma$ if $|\eta| < 1$. Since $\beta(B) \neq \alpha(B)$ is purely immaginary we have

$$\lim_{T \to +\infty} (1/T) \int_0^T e^{t\beta(B)} (1 + \eta \cos \alpha(B))\, dt = 0,$$

yielding the limit $\lim_{T \to +\infty} (1/T) \int_0^T A_\eta(t) = (\eta/2) A_\alpha$ (see [9] and [1], Lemma 2.3). Therefore $A_\alpha \neq 0$ belongs to Γ, hence $\exp t\,\mathrm{ad}(B) A_\alpha \in \Gamma$, $t \in \mathbb{R}$. Now,

14

$\exp \operatorname{tad}(B) A_\alpha = e^{t\alpha(B)} A_\alpha$, and since $\alpha(B) \neq 0$ we see that the complex subspace spanned by A_α is contained in Γ, that is, $\mathfrak{g}_\alpha \subset \Gamma$. The same way it follows that $\mathfrak{g}_{-\alpha} \subset \Gamma$, therefore $\mathfrak{g}(\alpha) \subset \Gamma$ implying that $G(\alpha) \subset \Gamma$ and $S = G$, by Theorem 3.1. $\qquad\square$

Remark: In [1] and [9] the above result is proved with the assumption that B is strong regular, which means that $\alpha(B) \neq 0$ for any root α and $\alpha(B) \neq \beta(B)$ for roots $\alpha \neq \beta$. With strong regularity it is possible to prove that $\mathfrak{g}(\alpha) \subset \Gamma$ for several roots α and conclude that $\Gamma = \mathfrak{g}$. By applying Theorem 3.1 it is enough to have $\mathfrak{g}(\alpha) \subset \Gamma$ for just one root α.

Theorem 4.2 *In the control system (1) take $B \in \mathfrak{h}$ with $B_{\mathrm{Re}} \neq 0$. Then the system is controllable in G if A and B generate $\mathfrak{g}$ (Lie algebra rank condition) and there exists a root α such that*

1. *$\operatorname{Im}\alpha(B) \neq 0$.*

2. *If $\beta \neq \alpha$ is a positive root such that $\operatorname{Re}\beta(B) \leq \operatorname{Re}\alpha(B)$ then $\operatorname{Re}\beta(B) < \operatorname{Re}\alpha(B)$.*

3. *$A_{\pm\alpha} \neq 0$ and $A_\gamma = 0$ in case $\operatorname{Re}\gamma(B) > \operatorname{Re}\alpha(B)$ or $\operatorname{Re}\gamma(B) < -\operatorname{Re}\alpha(B)$.*

Proof: For all $t \in \mathbb{R}$, $e^{\pm t \operatorname{Re}(\alpha(B))} \exp \operatorname{tad}(B) A \in \Gamma$. By the third condition

$$\exp \operatorname{tad}(B) A = \sum e^{t\gamma(B)} A_\beta$$

with the sum extended to γ with $-\alpha(B) \leq \gamma(B) \leq \alpha(B)$. But by the second condition

$$\lim_{t \to +\infty} e^{-t\operatorname{Re}\alpha(B)} \exp \operatorname{tad}(B) A = A_\alpha \quad \text{and} \quad \lim_{t \to -\infty} e^{t\operatorname{Re}\alpha(B)} \exp \operatorname{tad}(B) A = A_{-\alpha},$$

hence $A_{\pm\alpha} \in \Gamma$. Now, $\exp \operatorname{tad}(B) A_{\pm\alpha} = e^{\pm t\alpha(B)} A_{\pm\alpha}$, and since $\alpha(B) \neq 0$ we conclude that $\mathfrak{g}_{\pm\alpha} \subset \Gamma$. Hence $\mathfrak{g}(\alpha) \subset \Gamma$, $G(\alpha) \subset S$ and the result follows by Theorem 3.1. $\qquad\square$

Remark: In [1] and [9] the above theorem is proved by taking the highest root instead of an arbitrary root α. In fact, if $B_{\mathrm{Re}} \in \mathfrak{h}_{\mathbb{R}}^+$ (which can be assumed without loss of generality) then the second condition and part of the third condition are automatically true when α is the corresponding highest root. In this case the assumption in [1] and [9] is that $A_{\pm\alpha} \neq 0$. As to the first condition it follows if B is strong regular in the sense of [1] and [9]. This means that the dimension of $\ker \mathrm{ad}\,(B)$ is the rank of $\mathfrak{g}$ and the eigenvalues of the complexification $\mathrm{ad}\,(B)_{\mathbb{C}}$ of $\mathrm{ad}\,(B)$ are simple. In the complex Lie algebra $\mathfrak{g}$ we must complexify its realification. Then the eigenvalues of $\mathrm{ad}\,(B)_{\mathbb{C}}$ are those of $\mathrm{ad}\,(B)$ together with their complex conjugates. Hence the eigenvalues of $\mathrm{ad}\,(B)_{\mathbb{C}}$ are simple if and only if no eigenvalue of $\mathrm{ad}\,(B)$ is real. Therefore the strong regular condition implies that $\mathrm{Im}\beta\,(B) \neq 0$ for any root β.

References

[1] R. El Assoudi, J. P. Gauthier and I. Kupka, *On subsemigroups of semisimple Lie groups.* Annales de l'H. P., section 3, **13** (1996), 117-133.

[2] A. Borel, *Kählerian coset spaces of semi-simple Lie groups.* Proc. Nat. Acad. Sci. **40** (1954), 1147-1151.

[3] C. J. Braga Barros and L.A.B San Martin, *Controllability of Discrete-time Control Systems on the Symplectic Group.* Systems & Control Letters **42** (2001), 95-100.

[4] J.J. Duistermat, J.A.C. Kolk and V.S. Varadarajan, *Functions, flows and oscilatory integral on flag manifolds.* Compos. Math. 49, 309-398 (1983).

[5] T. Ferraiol, M. Patrão and L. Seco, *Jordan decomposition and dynamics on flag manifolds.* Discrete and Continuous Dynamical Systems, v. 26, p. 923-947, 2010.

[6] J.P. Gauthier, I. Kupka and G. Sallet, *Controllability of simple Lie groups right invariant systems on real.* Systems Control Letters **5** (1984) 187-190.

[7] Hilgert, J., K. H. Hofmann, and J. D. Lawson : "Lie Groups, Convex Cones and Semigroups,"Oxford University Press, 1989

[8] V. Jurdjevic, and I. Kupka, *Control systems subordinate to a group action: accessibility.* J. of Diff. Eq., **3**9 (1981), 186-211.

[9] V.Jurdjevic, and I. Kupka, *Control systems on semisimple Lie groups and their homogeneous spaces.* Ann.Inst. Fourier (Grenoble), **31**(1981), 151-179.

[10] A.W Knapp, Lie groups beyond and introduction. Second edition, Birkhauser (2004).

[11] O.G. do Rocio and L.A.B San Martin, *Connected components of open semigroups in semi-simple Lie groups.* Semigroup Forum **69** (2004), 1–29.

[12] L.A.B San Martin, *Invariant Control Sets on Flag Manifolds.* Math. of Control, Signal and Systems, vol. 6 (1993), 41-61.

[13] L.A.B San Martin, *Control Sets and Semigroups in Semi-Simple Lie Groups.* In Semigroups in Algebra, Analysis and Geometry. De Gruyter Expositions in Mathematics, vol. **20** (Editors: Hofmann, D. H. , Lawson, J. e Vinberg, E. B.) (1995), 275-291.

[14] L.A.B San Martin, *On global controllability of discrete-time control systems.* Math. Control Signals Systems, **8** (1995), 279-297.

[15] L.A.B San Martin and P.A Tonelli, *Semigroup actions on homogeneous spaces.* Semigroups Forum, **50** (1995), 59-88.

[16] L.A.B San Martin, *Order and Domains of Attraction of Control Sets in Flag Manifolds.* Journal of Lie Theory, **8** (1998), 335-350.

[17] L.A.B San Martin, *Maximal semigroups in semi-simple Lie groups.* Trans. Amer. Math. Soc., **353** (2001), 5165-5184.

[18] L.A.B San Martin and A. J. Santana, *Homotopy type of Lie semigroups in semi-simple Lie groups.* Monatshefte für Mathematik, **136** (2002), 151-173.

[19] L.A.B. San Martin and C.J.C. Negreiros, *Invariant almost Hermitian structures on flag manifolds.* Adv. Math., **178** (2003), 277-310.

[20] G. Warner, *Harmonic Analysis on Semi-simple Lie Groups.* Springer-Verlag, Berlin, (1972).